

SOME REMARKS ON GROUPS WITH ACTION ON ITSELF

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Abstract: We implement GAP functions about groups with action on itself and investigate some basic properties of small groups with action on itself of order < 32 .

Keywords: GAP, center, central series, ideal, nilpotency.

1 INTRODUCTION

The notion of group with action on itself (or shortly group with action) was introduced by T. Datuashvili in [1, 2] for solving the problem stated by J.-L. Loday in [5, 6], to define algebraic objects called “coquecigrues” which would have an analogous role for Leibniz algebras as groups have for Lie algebras. In [1], the notion of central series of groups with action was defined and given analogue of Witt’s construction [8] for such objects. For this a condition was found and constructed a subcategory of the category of groups with action whose objects satisfy this condition which was called Condition 1.

It was interesting to investigate what kind of properties do small groups with action on itself have. In standard GAP [3] library, there is no function about the groups with action on itself. For this first we added the function `AllGwAOnGroup(G)`, to obtain all group with action from a small group G . Then we added some functions to get ideals, center and isomorphism family of a given group with action GA . Finally, we added the functions `NilpotencyClassOfGwA(GA)`, `IsSingular(GA)`, `IsGwAC1(GA)` which check if the group with action GA is nilpotent, singular or satisfies Condition 1 or not. Additionally, to prove our main result Theorem 25, we constructed an enumeration table of group with action obtained from small groups with order < 32 .

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2 PRELIMINARIES

In this section, we will recall some definitions and properties of self acting groups from [1], [2]. Let G be a group which acts on itself from the right side; i.e we have a map $\varepsilon : G \times G \rightarrow G$ with

$$\begin{aligned}\varepsilon(g, g' + g'') &= \varepsilon(\varepsilon(g, g'), g''), \\ \varepsilon(g, 0) &= g, \\ \varepsilon(g' + g'', g) &= \varepsilon(g', g) + \varepsilon(g'', g), \\ \varepsilon(0, g) &= 0,\end{aligned}$$

for $g, g', g'' \in G$. Denote $\varepsilon(g, h) = g^h$, for $g, h \in G$. The pair $G^\bullet = (G, \varepsilon)$ is called a group with action. We denote the group operation additively, nevertheless the group is not abelian in general.

If (G', ε') is another group with action. A homomorphism $(G, \varepsilon) \longrightarrow (G', \varepsilon')$ is a group homomorphism $\varphi : G \longrightarrow G'$ for which the diagram

$$\begin{array}{ccc} G \times G & \xrightarrow{\varepsilon} & G \\ (\varphi, \varphi) \downarrow & & \downarrow \varphi \\ G' \times G' & \xrightarrow{\varepsilon'} & G' \end{array}$$

commutes. In other words, we have

$$\varphi(g^h) = \varphi(g)^{\varphi(h)}, \quad g, h \in G.$$

If we consider an action as a group homomorphism $G \xrightarrow{\nu} \text{Aut } G$, then a homomorphism between two groups with action means the commutativity of the diagram

$$\begin{array}{ccc} G & \xrightarrow{\nu} & \text{Aut } G \subset \text{Hom}(G, G) \\ \varphi \downarrow & & \downarrow \text{Hom}(G, \varphi) \\ G' & \xrightarrow{\nu'} & \text{Aut } G' \subset \text{Hom}(G', G') \\ & & \uparrow \text{Hom}(\varphi, G') \end{array}$$

so that $\varphi \cdot (\nu(h)) = \nu'(\varphi(h)) \cdot \varphi$, $h \in G$.

We shall denote the category of groups with action by \mathfrak{Gr}^\bullet . Let \mathfrak{Ab}^\bullet denote the category of abelian groups with action; here we mean the action within \mathfrak{Gr}^\bullet . We have the functors

$$\begin{array}{ccccc} & & & \xrightarrow{Q_1} & \\ & & & \xleftarrow{T} & \\ \mathfrak{Ab}^\bullet & \xrightleftharpoons[A]{E} & \mathfrak{Gr}^\bullet & \xrightleftharpoons[C]{Q_2} & \mathfrak{Gr} \end{array}$$

where $Q_1(G)$, for $G \in \mathfrak{Gr}^\bullet$, is the greatest quotient group of G which makes the action trivial; $Q_2(G)$ is a quotient of G by the equivalence relation generated by the relation $g^h \sim -h + g + h$, $g, h \in G$; A is the abelianization functor, thus $A(G) = G/(G, G)$, where (G, G) is the ideal of G generated by the commutator normal subgroup of G . $A(G)$ has the induced operation of action on itself. Each group can be considered as a group with the trivial action or with the action by conjugation, these give functors T and C , respectively. Every object of \mathfrak{Ab}^\bullet can be considered as an object of \mathfrak{Gr}^\bullet ; this functor is denoted by E . It is easy to see that the functors Q_1, Q_2 and A are left adjoints to the functors T, C and E respectively. Also we have the forgetful functor $\mathfrak{U} : \mathfrak{Gr}^\bullet \longrightarrow \mathfrak{Gr}$ which takes a group with action to its underlying group. In the sequel, the groups with actions will be denoted by the letters $G^\bullet = (G, \varepsilon_G), H^\bullet = (H, \varepsilon_H), \dots$

Now we will present some definitions and propositions from [1] without proofs.

Definition 1 Let $G^\bullet \in \mathfrak{Gr}^\bullet$. Let A be nonempty subset of G . If the conditions

- i. A is a normal subgroup of G as a group,
- ii. $a^g \in A$, for $a \in A, g \in G$,

iii. $-g + g^a \in A$, for $a \in A$ and $g \in G$,

satisfied, then A^\bullet is called an ideal of G^\bullet .

Thus an ideal of G^\bullet is a subobject of G^\bullet in \mathfrak{Gr}^\bullet . It is clear that G^\bullet itself and the trivial subobject of G^\bullet are ideals of G^\bullet . An intersection of any system of ideals of G^\bullet is an ideal, and therefore we conclude that there exists the ideal generated by a system of elements of G^\bullet .

Proposition 2 Let $G^\bullet \in \mathfrak{Gr}^\bullet$ and A^\bullet be an ideal of G^\bullet . For $a_1, a_2 \in A$, $g_1, g_2 \in G$ we have

$$(a_1 + g_1)^{a_2 + g_2} \in g_1^{g_2} + A.$$

Let A^\bullet and B^\bullet be subobjects of G^\bullet . Denote by $\{A, B\}$ the subobject of G^\bullet generated by A^\bullet and B^\bullet , and let $A + B$ denote the subset of G

$$A + B = \{a + b : a \in A, b \in B\}.$$

Proposition 3 If A^\bullet is an ideal of G^\bullet and B^\bullet is a subobject of G^\bullet , then

$$\{A, B\} = A + B$$

Proposition 4 If A^\bullet and B^\bullet are ideals of G^\bullet , then $A + B$ is also an ideal.

Proposition 5 If A^\bullet is an ideal of G^\bullet , then the quotient group G/A with the induced action on itself is an object of \mathfrak{Gr}^\bullet .

In what follows, for $G^\bullet \in \mathfrak{Gr}^\bullet$ and $g, g' \in G$, $[g, g']$ will indicate the element $-g + g^{g'}$ of G and (g, g') the commutator $-g - g' + g + g'$.

Definition 6 Let A^\bullet and B^\bullet be subobjects of G^\bullet . A commutator $[A, B]$ of A^\bullet and B^\bullet in G^\bullet is the ideal of $\{A, B\}$ generated by the elements

$$\{[a, b], [b, a], (a, b) : a \in A, b \in B\}$$

Definition 7 The (lower) central series

$$G^\bullet = G_1^\bullet \supset G_2^\bullet \supset \cdots \supset G_n^\bullet \supset G_{n+1}^\bullet \supset \cdots$$

of the object G^\bullet is defined inductively by

$$G_n^\bullet = [G_1, G_{n-1}] + [G_2, G_{n-2}] + \cdots [G_{n-1}, G_1].$$

Proposition 8 For each $n \geq 1$, G_{n+1}^\bullet is an ideal of G_n^\bullet .

Condition 1 For each $x, y, z \in G$,

$$x - x^{(z^x)} + x^{y+z^x} - x + x^z - x^{z+y^z} = 0.$$

In [1], category of groups (Abelian groups) with action $\mathfrak{Gr}^c(\mathfrak{Ab}^c)$ satisfying this condition and category of Lie-Leibniz algebras \mathfrak{LL} were defined. Then proved that the analogue of Witt's construction defines a functor $LL : \mathfrak{Gr}^c \longrightarrow \mathfrak{LL}$ which gives rise to Leibniz algebras (introduced in [5]) over the ring of integers by compositions $\mathfrak{Gr}^c \xrightarrow{A} \mathfrak{Ab}^c \xrightarrow{L} \mathfrak{Leibniz}$, $\mathfrak{Gr}^c \xrightarrow{LL} \mathfrak{LL} \xrightarrow{S_2} \mathfrak{Leibniz}$. The details of the functors can be found in [1].

Example 9 [1] Let G be a group. Consider G^\bullet as a group with action with the (right) action by conjugation. Then G^\bullet satisfies Condition 1.

Example 10 [1] Each group with the trivial action satisfies Condition 1.

Example 11 [1] Let G^\bullet be the abelian group of integers \mathbb{Z}^\bullet , which acts on itself in the following way: $x^y = (-1)^y x$. We have $[x, y] = 0$ for y even, $[x, y] = -2x$ for y odd and $G_n = 2^{n-1}\mathbb{Z}$. It is easy to see that \mathbb{Z}^\bullet satisfies Condition 1.

Example 12 Consider the group $C_2 \times C_2$ (Klein four). We diagramized the multiplication table of $C_2 \times C_2$ by

$+$	e	a	b	ab
e	e	a	b	ab
a	a	e	ab	b
b	b	ab	e	a
ab	ab	b	a	e

We have ten groups with action obtained from $C_2 \times C_2$. Four of them satisfy Condition 1. Whose which will be denoted by $(C_2 \times C_2, \varepsilon_i)$, $i = 1, 2, 3, 4$. Table of the actions ε_i are as follows;

ε_1	e	a	b	ab
e	e	a	b	ab
a	e	a	b	ab
b	e	a	ab	b
ab	e	a	ab	b

ε_2	e	a	b	ab
e	e	a	b	ab
a	e	a	b	ab
b	e	a	b	ab
ab	e	a	b	ab

ε_3	e	a	b	ab
e	e	a	b	ab
a	e	ab	b	a
b	e	a	b	ab
ab	e	ab	b	a

ε_4	e	a	b	ab
e	e	a	b	ab
a	e	b	a	ab
b	e	b	a	ab
ab	e	a	b	ab

where any object a_{ij} in the tables shows the right action of a_i on a_j .

3 CENTER AND NILPOTENCY OF A GROUP WITH ACTION

In this section first we introduce the center of a group with action and prove that our definition coincides with generalized definition of centers in algebraic categories introduced by Hug in [4]. Then we introduce the notion of nilpotency (class) of a group with action and give the main theorem which states that the nilpotent group with action obtained from group with order < 32 have nilpotency class 1 or 2.

Definition 13 Let $G^\bullet \in \mathfrak{Gr}^\bullet$. Then

$$\{g \in G : g + h = h + g, g = g^h, h = h^g, \text{ for all } h \in G\}$$

is called the center of G^\bullet and denoted by $Z(G^\bullet)$.

Proposition 14 Let $G^\bullet \in \mathfrak{Gr}^\bullet$. Then $Z(G^\bullet)$ is an ideal of G^\bullet .

Proof. Let $g, g' \in Z(G^\bullet)$. Since

$$\begin{aligned}
(g - g') + h &= g - g' + h \\
&= g - (-h + g') \\
&= g - (g' - h) \\
&= g + h - g' \\
&= h + (h - g'), \\
\\
g - g' &= g^h - g'^h \\
&= (g - g')^h, \\
\\
h^{(g - g')} &= h^{-(g' - g)} \\
&= (-h)^{g' - g} \\
&= -(h^{g' - g}) \\
&= ((h^{g'})^{-g}) \\
&= -(h^{-g}) \\
&= h^g \\
&= h,
\end{aligned}$$

for all $h \in G^\bullet$, we have $Z(G^\bullet)$ is a subobject of G^\bullet . On the other hand, we have $h + g - h = g$, for all $g \in Z(G^\bullet)$, $h \in G^\bullet$ which means $Z(G^\bullet)$ is a normal subgroup of G^\bullet as an additive group. Since $g^h = g$ and $-h + h^g = 0$, we have $g^h \in Z(G^\bullet)$ and $-h + h^g \in Z(G^\bullet)$ for all $g \in Z(G^\bullet)$, $h \in G^\bullet$. Consequently, $Z(G^\bullet)$ is an ideal of G^\bullet . ■

Definition 15 [4] Two coterminal morphisms $\beta_1 : B_1 \longrightarrow A$ and $\beta_2 : B_2 \longrightarrow A$ are said to commute if there exists a morphism

$$\beta_1 \circ \beta_2 : B_1 \times B_2 \longrightarrow A$$

making the diagram

$$\begin{array}{ccccc}
B_1 & \xrightarrow{\Gamma_1} & B_1 \times B_2 & \xleftarrow{\Gamma_2} & B_2 \\
& \searrow \beta_1 & \downarrow \beta_1 \circ \beta_2 & \swarrow \beta_2 & \\
& & A & &
\end{array}$$

commutative, where Γ_i for $i = 1, 2$ as usual denote the morphisms of the direct product. In particular, the morphism $\alpha : B \longrightarrow A$ said to be central if identity morphism on A commutes with α , i.e., if it makes the diagram

$$\begin{array}{ccccc}
A & \xrightarrow{\quad} & A \times B & \xleftarrow{\quad} & B \\
& \searrow 1_A & \downarrow & \swarrow \alpha & \\
& & A & &
\end{array}$$

commutative. In addition, if we have a monomorphism $\alpha : B \longrightarrow A$, then it is said that B is a central subobject of A .

Definition 16 [4] The center of an object is defined as the maximal central subobject, relative to the order relation that exists on the set of monomorphisms.

Proposition 17 Let $G^\bullet \in \mathfrak{Gr}^\bullet$. Then $Z(G^\bullet)$ is the maximal central subobject of G^\bullet .

Proof. Let H^\bullet be central subobject of G^\bullet . Then there exist a monomorphism $\alpha : H^\bullet \longrightarrow G^\bullet$ and an homomorphism $\beta : G^\bullet \times H^\bullet \longrightarrow G^\bullet$ which makes diagram

$$\begin{array}{ccccc} G^\bullet & \xrightarrow{\quad} & G^\bullet \times H^\bullet & \xleftarrow{\quad} & H^\bullet \\ & \searrow & \downarrow \beta & \swarrow \alpha & \\ & & G^\bullet & & \end{array}$$

commutative. So we have $\beta(g, 0) = g$, for all $g \in G$ and $\beta(0, h) = \alpha(h)$, for all $h \in H$.

Consequently, we have $\beta(g, h) = g + \alpha(h)$, for all $g \in G$, $h \in H$ which makes α a group with action homomorphism.

Indeed,

$$\begin{aligned} \alpha(h + h') &= \beta(0, h + h') \\ &= \beta((0, h) + (0, h')) \\ &= \beta(0, h) + \beta(0, h') \\ &= \alpha(h) + \alpha(h') \end{aligned}$$

and similarly, we have $\alpha \circ \varepsilon_H = \varepsilon_G \circ (\alpha, \alpha)$ as required.

Now, we will show that $\alpha(H^\bullet) \subseteq Z(G^\bullet)$.

Since

$$\begin{aligned} \alpha(h) + g &= \beta(0, h) + \beta(g, 0) \\ &= \beta(g, h) \\ &= \beta((g, 0) + (0, h)) \\ &= \beta(g, 0) + \beta(0, h) \\ &= g + \alpha(h), \end{aligned}$$

$$\begin{aligned} g^{\alpha(h)} &= (\beta(g, 0))^{\beta(0, h)} \\ &= \beta((g, 0)^{(0, h)}) \\ &= \beta(g^0, 0^h) \\ &= \beta(g, 0) \\ &= g \end{aligned}$$

and similarly $(\alpha(h))^g = \alpha(h)$, for all $g \in G$, $h \in H$, we have $\alpha(H^\bullet) \subseteq Z(G^\bullet)$, which means that $Z(G^\bullet)$ is the maximal central subobject, as required. ■

Corollary 18 Definition 13 coincides with Hug's definition given in [4], i.e. $Z(G^\bullet)$ is a maximal central subobject of G^\bullet in \mathfrak{Gr}^\bullet .

Proof. Follows from Definitions 13, 16 and Proposition 17. ■

Let $G^\bullet \in \mathfrak{Gr}^\bullet$. Obviously $Z(G^\bullet)$ is a normal subgroup of $Z(G)$. In addition, for any group H , we have the group with action H^\bullet whose underlying group is H and the action is defined by conjugation. In this case, $Z(H)$ coincides with $Z(H^\bullet)$.

The category \mathfrak{Gr}^\bullet is not a category of interest since the binary operation (i.e. the action) is not distributive. Nevertheless, we define the singular objects in \mathfrak{Gr}^\bullet in analogues way as it is in [7].

Definition 19 We say a group with action is singular if it coincides with its center.

Example 20 The Klein four group $C_2 \times C_2$ with the action ε_1 defined in Example 12 is singular.

A group with actions obtained from an abelian group is not a singular group with action, in general. The following GAP session gives two groups with action obtained from cyclic group C_4 . One of them is singular and the other is not.

```
gap> K := SmallGroup(4,1);; StructureDescription(G);
"C4"
gap> IsAbelian(K);
true
gap> aKA := AllGwAOnGroup(K);;
gap> List(aKA, i -> IsAbelianGwA(K));
[ true, false ]
```

Proposition 21 A group with action G^\bullet is singular iff

$$[G^\bullet, G^\bullet] = 0$$

Proof. Direct checking. ■

Definition 22 Let G^\bullet be a group with action. G^\bullet is called nilpotent if $G_n^\bullet = 0$ for some positive integer n in the lower central series of G^\bullet . The less $(n - 1)$ satisfying $G_n^\bullet = 0$, is called the nilpotency class of G^\bullet .

Corollary 23 Let $G^\bullet \in \mathfrak{Gr}^\bullet$. If G^\bullet is singular then G^\bullet is nilpotent with nilpotency class 1.

Proof. Follows from proposition 21. ■

Proposition 24 Let G^\bullet be a nilpotent group with action. If G^\bullet has nilpotency class 1 or 2, then G^\bullet satisfies Condition 1.

Proof. As it is proved in [1], Condition 1 can be expressed as

$$[x^y, [y; z]] = [[x, y], z^x] + [[x, z], y^z],$$

for all $x, y, z \in G^\bullet$, called as Condition 1', from which the required results obtained. ■

Theorem 25 Let $G^\bullet \in \mathfrak{Gr}^\bullet$ whose underlying group has order < 32 . If G^\bullet is nilpotent, then its nilpotency class is 1 or 2.

Proof. Follows from the table given in section 4. ■

Example 26 Not only the nilpotent groups with action, also some other groups with action which are not nilpotent satisfy Condition 1. The group with action G^\bullet obtained from

$$Q_8 = \langle a, b, c \rangle = \{e, a, b, c, ab, ac, bc, abc\}$$

quaternion group of order 8, whose action table is

ε	e	a	b	c	ab	ac	bc	abc
e	e	a	b	c	ab	ac	bc	abc
a	e	a	b	c	ab	ac	bc	abc
b	e	ac	abc	c	bc	a	ab	b
c	e	a	b	c	ab	ac	bc	abc
ab	e	ac	abc	c	bc	a	ab	b
ac	e	a	b	c	ab	ac	bc	abc
bc	e	ac	abc	c	bc	a	ab	b
abc	e	ac	abc	c	bc	a	ab	b

satisfies the Condition 1.

Remark 27 *There are 52 groups with action obtained from Q_8 . 36 of them are not nilpotent and 6 of these 36 satisfy Condition 1.*

4 GAP IMPLEMENTATIONS OF GROUPS WITH ACTION

There is no function for presenting of a group with action in standard GAP library. First, we add the function `AllGwAOnGroup(G)`, whose output is all groups with action obtained from a given group G , by constructing all homomorphisms from G to its automorphism group. For example, if we take $G = S_3$, we have the following;

```
gap> G := SmallGroup(6,1);
<pc group of size 6 with 2 generators>
gap> StructureDescription(G);
"S3"
gap> aGA := AllGwAOnGroup(G);
[ GroupWithAction [ Group( [ f1, f2 ] ), * ],
  GroupWithAction [ Group( [ f1, f2 ] ), * ],
  GroupWithAction [ Group( [ f1, f2 ] ), * ],
  GroupWithAction [ Group( [ f1, f2 ] ), * ],
  GroupWithAction [ Group( [ f1, f2 ] ), * ],
  GroupWithAction [ Group( [ f1, f2 ] ), * ],
  GroupWithAction [ Group( [ f1, f2 ] ), * ],
  GroupWithAction [ Group( [ f1, f2 ] ), * ],
  GroupWithAction [ Group( [ f1, f2 ] ), * ],
  GroupWithAction [ Group( [ f1, f2 ] ), * ] ]
gap> Length(aGA);
10
```

We add the function `IsGwA()` implemented for testing is the given structure is a group with action or not.

```
gap> IsGwA(G);
false
gap> GA := aGA[2];;
gap> IsGwA(GA);
true
gap> List(aGA, i -> IsGwA(i));
[ true, true, true, true, true, true, true, true, true, true ]
```

We add the function `IsIdeal(GA,HA)`. By this we investigate if the subobject HA is an ideal of GA or not. In addition, the functions `NrIdealOnGwA(GA)`, `AllIdealOnGwA(GA)` are used to get all ideals and number of all ideals of a given group with action GA , respectively.

```
gap> AllIdealOnGwA(GA);
[ GroupWithAction [ Group( [ f1, f2 ] ), * ],
  GroupWithAction [ Group( [ f2 ] ), * ],
  GroupWithAction [ Group( <identity> of ... ), * ] ]
gap> HA := last[2];
GroupWithAction [ Group( [ f2 ] ), * ]
gap> IsIdeal(GA,HA);
true
```



```
gap> List(aGA,i -> NrIdealOnGwA(i));
[ 3, 3, 3, 3, 3, 3, 3, 3, 3, 3 ]
```

We add the function `NilpotencyClassOfGwA(GA)` to obtain the nilpotency class of a given group with action `GA`. The step-by-step construction of the function is as follows:

Step 1 : We add the `Commutator()` function to obtain the commutator subobject of a given group with action `GA`.

Step 2 : We obtain the lower central series of `GA` by the function `LowerCentralSeriesOfGwA(GA)`.

```
gap> Commutator(GA,HA);
GroupWithAction [ Group( [ <identity> of ..., f2, f2^2 ] ), * ]
gap> IsIdeal(GA,last);
true
gap> LowerCentralSeriesOfGwA(GA);
[ GroupWithAction [ Group( [ f1, f2 ] ), * ],
  GroupWithAction [ Group( [ <identity> of ..., f2, f2^2 ] ), * ] ]
gap> IsNilpotent(GA);
false
gap> NilpotencyClassOfGwA(GA);
0
gap> List(aGA,i -> IsNilpotent(i));
[ false, false, false, false, false, false, false, false, false, false ]
```

Then we add the function `CenterOfGwA(GA)` to obtain the center of a group with action `GA` and add the function `IsSingularGwA(GA)` in order to see if the group with action `GA` is singular or not.

```
gap> CA := CenterOfGwA(GA);
GroupWithAction [ Group( [ <identity> of ... ] ), * ]
gap> IsIdeal(GA,CA);
true
gap> IsAbelianGwA(GA);
false
gap> IsAbelianGwA(CA);
true
gap> List(aGA, i -> IsIdeal(i,CenterOfGwA(i)));
[ true, true, true, true, true, true, true, true, true, true ]
```

We use the following steps to obtain the isomorphism classes of a group with action;

Step 1 : We add the functions `GwAMorphism(phi)`, `IsGwAMorphism(phi)` to obtain and control group with action homomorphisms.

Step 2 : We add the function `IsIsomorphicGwA(GA,HA)` to check if two groups with action are isomorphic or not.

Step 3 : `IsomorphicGwAFamily(aGA,GA)` gives the isomorphism family of a given group with action.

```
gap> IsIsomorphicGwA(GA,HA);
false
gap> IsomorphicGwAFamily(aGA[1],aGA);
1 => [ 1 ]
gap> IsomorphicGwAFamily(aGA[2],aGA);
```

```

3 => [ 2, 3, 4 ]
gap> IsomorphicGwAFamily(aGA[5],aGA);
2 => [ 5, 10 ]
gap> IsomorphicGwAFamily(aGA[6],aGA);
3 => [ 6, 8, 9 ]
gap> IsomorphicGwAFamily(aGA[7],aGA);
1 => [ 7 ]

```

We add the function `IsGwAC1(GA)` which controls if the group with action `GA` satisfies Condition 1 or not.

```

gap> IsGwAC1(aGA[4]);
true
gap> List(aGA, i -> IsGwAC1(i));
[ true, true, true, true, false, false, true, false, false, false ]

```

We add the function `ActionTableOfGwA(GA)`, for constructing an action table of the group with action `GA`,

```

gap> MultiplicationTableOfGwA(aGA[4]);
[ [ <identity> of ..., f1, f2, f1*f2, f2^2, f1*f2^2 ],
  [ <identity> of ..., f1, f2^2, f1*f2^2, f2, f1*f2 ],
  [ <identity> of ..., f1, f2, f1*f2, f2^2, f1*f2^2 ],
  [ <identity> of ..., f1, f2^2, f1*f2^2, f2, f1*f2 ],
  [ <identity> of ..., f1, f2, f1*f2, f2^2, f1*f2^2 ],
  [ <identity> of ..., f1, f2^2, f1*f2^2, f2, f1*f2 ] ]

```

In this session, we select the fourth group with action obtained from $S_3 = \langle a, b \rangle = \{e, a, b, ab, b^2, ab^2\}$ whose action table is as follows:

ε	e	a	b	ab	b^2	ab^2
e	e	a	b	ab	b^2	ab^2
a	e	a	b^2	ab^2	b	ab
b	e	a	b	ab	b^2	ab^2
ab	e	a	b^2	ab^2	b	ab
b^2	e	a	b	ab	b^2	ab^2
ab^2	e	a	b^2	ab^2	b	ab

Using above implementations, we have following table which gives some algebraic properties of groups with action obtained from groups with order < 32 .

In the fifth row of the table the groups with action obtained from the Klein four group, $C_2 \times C_2$ are investigated. There are 10 groups with action and 3 isomorphism families. Two of them satisfy Condition 1 and one of them does not. One of them has 5 ideals and two of them have 3 ideals. There is one group with action with nilpotency class 2, one with nilpotency class 1 and one is not nilpotent.

GAP id	Name	$ Gr^\bullet $	$ Gr^\bullet/\sim $	$ GrC1^\bullet/\sim $	$ Ideals(Gr^\bullet/\sim) $	$ Ni(Gr^\bullet/\sim) $
[1, 1]	I	1	1	1	[1, 1]	[0, 1]
[2, 1]	C2	1	1	1	[2, 1]	[1, 1]
[3, 1]	C3	1	1	1	[2, 1]	[1, 1]

GAP id	Name	$ \text{Gr}^\bullet $	$ \text{Gr}^\bullet/\sim $	$ \text{GrC1}^\bullet/\sim $	$ \text{Ideals}(\text{Gr}^\bullet/\sim) $	$ \text{Ni}(\text{Gr}^\bullet/\sim) $
[4, 1]	C4	2	2	2	[3, 2]	[1, 1], [2, 1]
[4, 2]	C2xC2	10	3	2	[3, 2], [5, 1]	[0, 1], [1, 1], [2, 1]
[5, 1]	C5	1	1	1	[2, 1]	[1, 1]
[6, 1]	S3	10	5	3	[3, 5]	[0, 5]
[6, 2]	C6	2	2	2	[3, 1], [4, 1]	[0, 1], [1, 1]
[7, 1]	C7	1	1	1	[2, 1]	[1, 1]
[8, 1]	C8	4	4	4	[4, 4]	[0, 2], [1, 1], [2, 1]
[8, 2]	C4xC2	32	15	11	[4, 2], [5, 2], [6, 9], [8, 2]	[0, 4], [1, 1], [2, 10]
[8, 3]	D8	36	16	11	[4, 6], [6, 10]	[0, 6], [2, 10]
[8, 4]	Q8	52	10	7	[4, 4], [6, 6]	[0, 4], [2, 6]
[8, 5]	C2xC2xC2	736	14	6	[3, 1], [4, 2], [5, 1], [6, 7], [7, 1], [8, 1], [16, 1]	[0, 8], [1, 1], [2, 5]
[9, 1]	C9	3	2	2	[3, 2]	[1, 1], [2, 1]
[9, 2]	C3xC3	33	3	2	[3, 2], [6, 1]	[0, 1], [1, 1], [2, 1]
[10, 1]	D10	26	7	3	[3, 7]	[0, 7]
[10, 2]	C10	2	2	2	[3, 1], [4, 1]	[0, 1], [1, 1]
[11, 1]	C11	1	1	1	[2, 1]	[1, 1]
[12, 1]	C3:C4	20	10	6	[5, 10]	[0, 10]
[12, 2]	C12	4	4	4	[5, 2], [6, 2]	[0, 2], [1, 1], [2, 1]
[12, 3]	A4	33	8	4	[3, 8]	[0, 8]
[12, 4]	D12	64	19	7	[4, 3], [5, 10], [6, 1], [7, 5]	[0, 19]
[12, 5]	C6xC2	48	11	5	[2, 1], [3, 1], [4, 3], [5, 2], [6, 2], [7, 1], [10, 1]	[0, 9], [1, 1], [2, 1]
[13, 1]	C13	1	1	1	[2, 1]	[1, 1]
[14, 1]	D14	50	9	3	[3, 9]	[0, 9]
[14, 2]	C14	2	2	2	[3, 1], [4, 1]	[0, 1], [1, 1]
[15, 1]	C15	1	1	1	[4, 1]	[1, 1]
[16, 1]	C16	8	6	5	[5, 6]	[0, 3], [1, 1], [2, 2]
[16, 2]	C4xC4	832	73	54	[4, 4], [5, 7], [6, 6], [7, 4], [9, 33], [11, 16], [13, 2], [15, 1]	[0, 20], [1, 1], [2, 52]
[16, 3]	(C4xC2):C2	640	168	138	[4, 12], [5, 9], [6, 6], [7, 5], [9, 116], [11, 20]	[0, 32], [2, 136]
[16, 4]	C4:C4	448	161	138	[5, 3], [6, 8], [7, 8], [8, 4], [9, 118], [11, 20]	[0, 25], [2, 136]
[16, 5]	C8xC2	128	56	40	[5, 4], [6, 4], [7, 26], [9, 18], [11, 4]	[0, 44], [1, 1], [2, 11]
[16, 6]	C8:C2	128	56	40	[5, 4], [6, 8], [7, 24], [9, 20]	[0, 46], [2, 10]
[16, 7]	D16	256	63	45	[5, 13], [7, 50]	[0, 63]
[16, 8]	QD16	144	88	80	[7, 88]	[0, 88]

GAP id	Name	$ Gr^\bullet $	$ Gr^\bullet/\sim $	$ GrC1^\bullet/\sim $	$ Ideals(Gr^\bullet/\sim) $	$ Ni(Gr^\bullet/\sim) $
[16, 9]	Q16	192	57	45	[5, 7], [7, 50]	[0, 57]
[16, 10]	C4xC2xC2	14912	404	105	[4, 7], [5, 10], [6, 6], [7, 129], [8, 92], [9, 75], [10, 11], [11, 31], [13, 4], [17, 28], [19, 9], [27, 2]	[0, 300], [1, 1], [2, 103]
[16, 11]	C2xD8	7744	578	166	[4, 14], [5, 25], [6, 16], [7, 157], [8, 149], [9, 113], [10, 2], [11, 16], [17, 76], [19, 10]	[0, 426], [2, 152]
[16, 12]	C2 x Q8	9536	275	80	[4, 4], [5, 14], [6, 14], [7, 67], [8, 77], [9, 51], [10, 2], [11, 8], [17, 32], [19, 6]	[0, 209], [2, 66]
[16, 13]	(C4xC2):C2	1856	232	128	[7, 40], [8, 64], [9, 24], [17, 104]	[0, 128], [2, 104]
[17, 1]	C17	1	1	1	[2, 1]	[1, 1]
[18, 1]	D18	82	14	3	[4, 14]	[0, 14]
[18, 2]	C16	6	4	3	[4, 2], [6, 2]	[0, 2], [1, 1], [2, 1]
[18, 3]	C3xS3	24	12	7	[4, 1], [5, 6], [6, 5]	[0, 12]
[18, 4]	(C3xC3):C2	4510	41	6	[3, 4], [4, 23], [5, 9], [7, 5]	[0, 41]
[18, 5]	C6xC3	78	7	4	[4, 2], [6, 3], [7, 1], [12, 1]	[0, 5], [1, 1], [2, 1]
[19, 1]	C19	1	1	1	[2, 1]	[1, 1]
[20, 1]	Q20	72	16	7	[4, 2], [5, 14]	[0, 16]
[20, 2]	C20	8	6	5	[4, 2], [5, 2], [6, 2]	[0, 4], [1, 1], [2, 1]
[20, 3]	C5:C4	36	9	5	[4, 9]	[0, 9]
[20, 4]	D20	144	25	7	[4, 3], [5, 14], [6, 1], [7, 7]	[0, 25]
[20, 5]	C10xC2	40	9	4	[4, 3], [5, 2], [6, 2], [7, 1], [10, 1]	[0, 7], [1, 1], [2, 1]
[21, 1]	C7:C3	57	10	4	[3, 10]	[0, 10]
[21, 2]	C21	3	2	2	[3, 1], [4, 1]	[0, 1], [1, 1]
[22, 1]	D22	122	13	3	[3, 13]	[0, 13]
[22, 2]	C22	2	2	2	[3, 1], [4, 1]	[0, 1], [1, 1]
[23, 1]	C23	1	1	1	[2, 1]	[1, 1]
[24, 1]	C3:C8	40	20	12	[7, 20]	[0, 20]
[24, 2]	C24	8	8	8	[7, 4], [8, 4]	[0, 6], [1, 1], [2, 1]
[24, 3]	SL(2,3)	33	8	2	[4, 8]	[0, 8]
[24, 4]	C3:Q8	448	92	49	[6, 6], [7, 20], [8, 16], [9, 50]	[0, 92]
[24, 5]	C4xS3	256	112	80	[8, 28], [9, 70], [10, 4], [11, 10]	[0, 112]
[24, 6]	D24	576	106	49	[6, 10], [7, 30], [8, 16], [9, 50]	[0, 106]

GAP id	Name	$ \text{Gr}^\bullet $	$ \text{Gr}^\bullet/\sim $	$ \text{GrC1}^\bullet/\sim $	$ \text{Ideals}(\text{Gr}^\bullet/\sim) $	$ \text{Ni}(\text{Gr}^\bullet/\sim) $
[24, 7]	C2x(C3:C4)	512	98	49	[6, 2], [7, 14], [8, 15], [9, 50], [10, 2], [11, 5], [13, 10]	[0, 98]
[24, 8]	(C6xC2):C2	256	112	80	[8, 32], [9, 80]	[0, 112]
[24, 9]	C12xC2	128	53	38	[6, 2], [7, 6], [8, 3], [9, 24], [10, 2], [11, 3], [12, 9], [13, 2], [16, 2]	[0, 42], [1, 1], [2, 10]
[24, 10]	C3xD8	144	58	38	[6, 10], [7, 6], [8, 6], [9, 26], [12, 10]	[0, 48], [2, 10]
[24, 11]	C3xQ8	240	32	18	[3, 1], [4, 1], [6, 6], [7, 4], [8, 4], [9, 10], [12, 6]	[0, 26], [2, 6]
[24, 12]	S4	58	11	4	[4, 11]	[0, 11]
[24, 13]	C2xA4	42	10	6	[4, 1], [5, 1], [6, 8]	[0, 10]
[24, 14]	C2xC2xS3	7168	196	46	[3, 4], [4, 7], [5, 8], [6, 4], [7, 39], [8, 28], [9, 71], [10, 9], [11, 15], [13, 5], [18, 1], [21, 5]	[0, 196]
[24, 15]	C6xC2xC2	6636	69	17	[3, 1], [4, 4], [5, 4], [6, 6], [7, 16], [8, 5], [9, 15], [10, 3], [11, 3], [12, 7], [13, 1], [14, 1], [16, 1], [21, 1], [32, 1]	[0, 63], [1, 1], [2, 5]
[25, 1]	C25	5	2	2	[3, 2]	[1, 1], [2, 1]
[25, 2]	C5xC5	145	3	2	[3, 2], [8, 1]	[0, 1], [1, 1], [2, 1]
[26, 1]	D26	170	15	3	[3, 15]	[0, 15]
[26, 2]	C26	2	2	2	[3, 1], [4, 1]	[0, 1], [1, 1]
[27, 1]	C27	9	3	2	[4, 3]	[0, 1], [1, 1], [2, 1]
[27, 2]	C9xC3	297	31	16	[4, 14], [5, 3], [7, 12], [10, 2]	[0, 17], [1, 1], [2, 13]
[27, 3]	(C3xC3):C3	2673	35	15	[4, 23], [7, 12]	[0, 23], [2, 12]
[27, 4]	C9:C3	297	48	27	[4, 27], [7, 21]	[0, 27], [2, 21]
[28, 1]	C7:C4	100	18	6	[5, 18]	[0, 18]
[28, 2]	C28	4	4	4	[5, 2], [6, 2]	[0, 2], [1, 1], [2, 1]
[28, 3]	D28	256	31	7	[4, 3], [5, 18], [6, 1], [7, 9]	[0, 31]
[28, 4]	C14xC2	40	9	4	[4, 3], [5, 2], [6, 2], [7, 1], [10, 1]	[0, 7], [1, 1], [2, 1]
[29, 1]	C29	1	1	1	[2, 1]	[1, 1]
[30, 1]	C5xS3	20	10	6	[5, 5], [6, 5]	[0, 10]
[30, 2]	C3xD10	52	14	6	[5, 7], [6, 7]	[0, 14]
[30, 3]	D30	260	35	9	[5, 35]	[0, 35]
[30, 4]	C30	4	4	4	[5, 1], [6, 2], [8, 1]	[0, 3], [1, 1]
[31, 1]	C31	1	1	1	[2, 1]	[1, 1]

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